Bremsstrahlung by a rapidly accelerating point charge

Daniel Egaña and Iván Schmidt
Department of Physics, Universidad Técnica Federico Santa María, Valparaíso, Chile;
Center of Subatomic Physics, Valparaíso, Chile;
and Valparaíso Center for Science and Technology, Valparaíso, Chile
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The classical problem of radiation emitted by an accelerated charge in the relativistic case is considered. Only accelerations which occur in a short time interval are taken into account. We first discuss the one dimensional trajectories, for later extending the analysis to the two-dimensional case by means of a first-order approximation. We find an expression for the allowed radiated frequencies and we discuss the inconvenience of the concept of formation time of radiation. The results allow us to treat the case of successive collisions, in order to study the behavior of the interference terms.

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I. INTRODUCTION

The classical problem of the radiation produced by an accelerated charge in covariant formulation is vastly treated in sources as [1,2]. These treatments are interesting not only because they agree with the quantum-mechanical results in the soft-photon limit, but also because they provide useful information for understanding some radiation phenomena.

Nevertheless, the treatment of some of the simplest radiation processes is not a closed subject, mainly because of mathematical difficulties. In particular, the radiation emitted, when a point charge collides in a known space-time coordinate with fixed initial and final velocities, is frequently treated in the non relativistic case as in [3] or in the one dimensional relativistic case [4], where the direction of the velocity of the particle does not change during the collision, but there is no specific information about the analysis in the two-dimensional relativistic regime, which is the most general case. The aim of this paper is precisely to obtain a simple classical expression for the radiated energy in this general case, where the direction of the initial and final velocities of the particle can be different. It will be seen that this expression provides a deeper insight in the form and characteristics of the soft-photon radiation produced in single and multiple collisions.

We know from the analysis of one dimensional trajectories that the small but finite time of the collision induces a cutoff in the radiated frequencies. This cutoff and its dependency on the angle of radiation is well known for the one-dimensional (1D) case, but in the two-dimensional case, mathematical difficulties arise and therefore the cutoff is frequently treated as a constant [5].

In this paper we find the explicit dependence of this cutoff on the angle of radiation and the initial and final velocities of the particle in two-dimensional case. In order to obtain these results, we exploit the fact that the collision occurs in a very short time interval and therefore a first-order approximation in the proper time is well suited. This treatment allows us to find a simple expression for the angular distribution of radiated energy, and it gives us important information about the interference terms in the case of successive collisions.

The analysis leads us to discuss the inconvenience of the concept of formation length or formation time of radiation in classical electrodynamics, because it could lead to confusions.

II. CLASSICAL THEORY OF BREMSSTRAHLUNG

When a charged point particle undergoes an acceleration in a short time interval, its stationary field is distorted and the energy transferred in the collision process is radiated. After the particle leaves the acceleration zone with constant velocity, it recovers its normal stationary field. It is possible to find the spectral decomposition of the radiated energy by using the Fourier transform of the current [5], namely,

$$dE = -\frac{\tilde{j}^{\mu}(k)\tilde{j}^{*}_{\mu}(k)}{2(2\pi)^3}d^3\mathbf{k}, \tag{1}$$

where Heaviside-Lorentz units are been used with $c=1$, and $k=(\omega,\mathbf{k})$ is a lightlike four-vector. In terms of the world line $x(\tau)$, the Fourier transform of the current for a point charge is given by

$$\tilde{j}^{\mu}(k) = e\int d\tau \frac{dx^{\mu}}{d\tau}\exp[ik\cdot x(\tau)]. \tag{2}$$

The calculation of the Fourier transform needs information of the current in infinity. But taking into account that the radiation process takes place in a localized region in space-time, the different border conditions in infinity should not affect the calculation of the radiated energy.

In the case of an instantaneous acceleration in the origin of the space-time coordinate system, we have

$$x^{\mu}(\tau) = \begin{cases} v_1^{\mu}\tau, & \tau < 0 \\ v_2^{\mu}\tau, & \tau > 0, \end{cases} \tag{3}$$

where $v_1^{\mu}=\cosh\,\eta_1(1,\mathbf{v_1})$ and $v_2^{\mu}=\cosh\,\eta_2(1,\mathbf{v_2})$ are arbitrary initial and final four velocities with rapidities η_1 and η_2. Then, the Fourier transform of the four current is given by

$$\tilde{j}^{\mu}(k) = -ie\left(\frac{v_1^{\mu}}{k\cdot v_1} - \frac{v_2^{\mu}}{k\cdot v_2}\right). \tag{4}$$

If we use this expression for finding the total radiated energy integrating Eq. (1), we realize that this energy is infinite. This divergence is due to the fact that an instantaneous change in velocity is only produced by an infinite sudden acceleration, which is of course an unrealistic model of the physical process. This model is treated in detail in [5].

In order to avoid this divergence, we proceed directly to a more realistic model, in which the particle changes its velocity gradually through a finite proper acceleration a_0. The particle experiences this acceleration only for a short time interval $\Delta\tau \propto 1/a_0$ in a definite region of space-time, and the radiation is as usual measured by a distant observer. The interesting fact is that the results of this last model are tightly related to the sudden acceleration model. In the one-dimensional case, the corresponding four current of such a trajectory is the same as the one given in Eq. (4), with the only difference that a cutoff in frequency is introduced ([4]), given by $\omega \leq a_0/\sin(\theta)$, where θ is the angle between the axis in which the particle moves and the wave vector $\mathbf{k}$ (in the 1D case we have azimuthal symmetry). This cutoff in frequency may be interpreted by thinking that only the fast components of the Fourier transform of the field can follow the charge in its change in direction, but the slow components are unable to remain attached to the particle and therefore they are radiated.

Following the conclusions of the one dimensional case, to find the radiated energy in the two-dimensional case, the four current of an instantaneously accelerated charge should be used with a specific cutoff in frequency. This cutoff is a mathematical consequence of the finite time interval in which the acceleration is considerable, and can be explained and found as follows. The inner product $k\cdot x(\tau)$ depends on the polar angle θ, the azimuthal angle ϕ, the angle α between the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$, and the proper time τ. These three angles will be denoted by the variable $\chi=(\theta,\phi,\alpha)$. Then we have

$$k\cdot x(\tau) = \omega[t(\tau) - \hat{\mathbf{k}}\cdot\mathbf{x}(\tau)] = \omega f(\chi,\tau). \tag{5}$$

Now, let $F(\omega',\omega,\chi)$ be the Fourier transform of the function $\exp[i\omega f(\chi,\tau)]$. Then, it is possible to express the four current given in Eq. (2) in terms of this Fourier transform and the four acceleration using integration by parts, namely,

$$\tilde{j}^{\mu}(k) = ie\int_{-\infty}^{\infty} d\omega' \frac{F(\omega',\omega,\chi)}{\omega'}\int_{-\infty}^{\infty} d\tau \frac{d^2x^{\mu}}{d\tau^2}\exp(i\omega'\tau). \tag{6}$$

Notice that the boundary terms were discarded because they do not contribute to the radiation produced by the acceleration of the charge, as discussed previously. The last integral is just the Fourier transform of the four acceleration. If we consider that this acceleration is considerable only during a time interval $\Delta\tau$, and that the acceleration is a smooth function, then its Fourier transform has frequency content only in the range $\omega' \leq 1/\Delta\tau$. In this time interval we can approximate $\exp(i\omega'\tau)\approx 1$. This assumptions yield

$$\tilde{j}^{\mu}(k) = ie(v_2^{\mu} - v_1^{\mu})\int_{-\infty}^{\infty} d\omega' \frac{F(\omega',\omega,\chi)}{\omega'}, \quad \omega' \leq 1/\Delta\tau. \tag{7}$$

Besides, the first-order approximation of $f(\chi,\tau) = c_1(\chi)\tau + O(\tau^2)$ allows us to obtain an analytic expression for the Fourier transform,

$$F(\omega',\omega,\chi) = \delta(\omega' - \omega c_1(\chi)), \tag{8}$$

therefore, we find that the four current and its cutoff in frequency are given by

$$\tilde{j}^{\mu}(k) = \frac{ie(v_2^{\mu} - v_1^{\mu})}{\omega c_1(\chi)}, \quad \omega c_1(\chi) \leq 1/\Delta\tau. \tag{9}$$

Of these two approximations, we only need the approximation for the cutoff in frequency, because as noted, we will use the four current for the instantaneously accelerated charge. The first-order approximation for the four current can also be used and the results are similar.

It is necessary to clarify the meaning of the coefficient $c_1(\chi)$ and the validity of the first-order approximation. By comparing with Eq. (5) we find that

$$\omega c_1(\chi) = k\cdot \left.\frac{dx(\tau)}{d\tau}\right|_{\tau=0}. \tag{10}$$

The value of the four velocity evaluated in $\tau=0$ is meaningful and unambiguous only if we consider that a first-order approximation is equivalent to state that the particle moves with a constant velocity in the collision process. This velocity can be taken to be the average velocity, simply given by

$$\left.\frac{dx^{\mu}(\tau)}{d\tau}\right|_{\tau=0} = \frac{v_1^{\mu} + v_2^{\mu}}{2}. \tag{11}$$

As a consequence, the approximation is valid only if the particle has four velocities similar to the average during the collision process. This is not the case in the ultrarelativistic case, where the initial and final four velocities can be very different. Anyway, it should be noted that in this cases, the approximation can always be made in a reference system where the particle has lower velocities by using the fact that the energy is a Lorentz invariant.

Now, by using Eqs. (9) and (10) we find the explicit expression for the cutoff,

$$\omega \leq \frac{2}{\Delta\tau[\cosh(\eta_1) + \cosh(\eta_2) - \sinh(\eta_1)\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_{\mathbf{1}} - \sinh(\eta_2)\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_{\mathbf{2}}]} = \xi(\chi, v_1, v_2), \tag{12}$$

where the particle initial and final rapidities η_1 and η_2 are introduced by using the expressions for the Lorentz factors $\gamma(v_1)=\cosh\,\eta_1$, $\gamma(v_2)=\cosh\,\eta_2$. This expression indicates the dependency of the cutoff on the initial and final rapidities

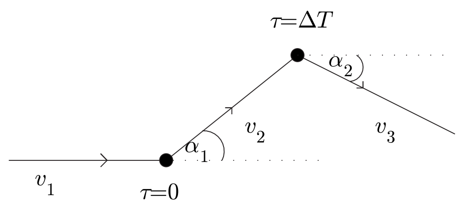

FIG. 1. Simplified scheme of a double collision of the point charge.

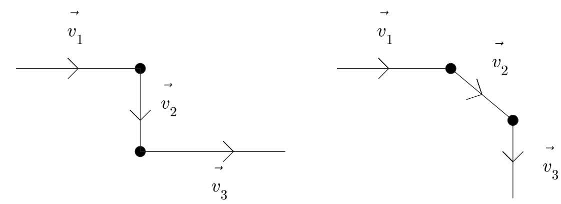

FIG. 2. Paradigm of double collisions with $A>0$ and $A<0$ respectively.

and with the angle between the wave vector and the velocities. The spectral decomposition of the radiated energy is simply

$$\frac{dE}{d^3\mathbf{k}}=\frac{e^2}{2(2\pi)^3}\left(\frac{2(v_1\cdot v_2)}{(k\cdot v_1)(k\cdot v_2)}-\frac{1}{(k\cdot v_1)^2}-\frac{1}{(k\cdot v_2)^2}\right), \tag{13}$$

where $\omega\leq\xi$. In order to obtain the angular distribution of radiated energy, we must perform the integration in ω, which is straightforward, yielding

$$\frac{dE}{d\Omega}=\frac{e^2\xi(\chi,v_1,v_2)}{2(2\pi)^3}\left(\frac{2(1-\mathbf{v_1}\cdot\mathbf{v_2})}{(1-\hat{\mathbf{k}}\cdot\mathbf{v_1})(1-\hat{\mathbf{k}}\cdot\mathbf{v_2})}-\frac{1}{\cosh(\eta_1)^2(1-\hat{\mathbf{k}}\cdot\mathbf{v_1})^2}-\frac{1}{\cosh(\eta_2)^2(1-\hat{\mathbf{k}}\cdot\mathbf{v_2})^2}\right). \tag{14}$$

Note that if we invert the order of the velocities v_1 and v_2, the radiated energy remains the same.

III. SUCCESSIVE ACCELERATIONS

If we have a second acceleration after a proper time $\Delta T\gg\Delta\tau$ as depicted in Fig. 1, we have two contributions to the current, given by

$$\tilde{g}^{\mu}(k)=-ie\left(\frac{v_1^{\mu}}{k\cdot v_1}-\frac{v_2^{\mu}}{k\cdot v_2}\right),$$

$$\tilde{h}^{\mu}(k)=-ie\left(\frac{v_2^{\mu}}{k\cdot v_2}-\frac{v_3^{\mu}}{k\cdot v_3}\right)\exp(ik\cdot v_2\Delta T), \tag{15}$$

where the cutoff in frequency for $\tilde{g}^{\mu}$ and $\tilde{h}^{\mu}$ are given by $\omega\leq\xi_1(\chi_1,v_1,v_2)$ and $\omega\leq\xi_2(\chi_2,v_2,v_3)$, respectively. The total current is given by $\tilde{j}^{\mu}(k)=\tilde{g}^{\mu}(k)+\tilde{h}^{\mu}(k)$. The radiated energy is easily obtained by means of Eq. (1),

$$\frac{dE}{d^3\mathbf{k}}=\frac{dE_1}{d^3\mathbf{k}}+\frac{dE_2}{d^3\mathbf{k}}-A\frac{e^2\cos(\omega\delta)}{(2\pi)^3\omega^2}, \tag{16}$$

where $\delta=\gamma(v_2)(1-\hat{\mathbf{k}}\cdot\mathbf{v_2})\Delta T$ and γ is the Lorentz factor. The first and second terms on the right are just the contribution to the radiated energy given by the two accelerations separately. Therefore, for these terms we must take into account the cutoff function for the first and second collision, respectively, given by the functions ξ_1 and ξ_2. The remaining term represents the interference of the radiation. For this term, we must of course consider only frequencies under the minimum cutoff function, $\omega\leq\min(\xi_1,\xi_2)$. The factor A is a function of the velocities and the angles, given in Eq. (18).

Integrating in ω, we obtain the angular distribution of radiated energy

$$\frac{dE}{d\Omega}=\frac{dE_1}{d\Omega}+\frac{dE_2}{d\Omega}-A\frac{e^2\min(\xi_1,\xi_2)}{(2\pi)^3}\left(\frac{\sin(\min(\xi_1,\xi_2)\delta)}{\min(\xi_1,\xi_2)\delta}\right). \tag{17}$$

$$A(\mathbf{v_1},\mathbf{v_2},\mathbf{v_3},\theta,\phi)=\left(\frac{(1-\mathbf{v_1}\cdot\mathbf{v_2})}{(1-\hat{\mathbf{k}}\cdot\mathbf{v_1})(1-\hat{\mathbf{k}}\cdot\mathbf{v_2})}+\frac{(1-\mathbf{v_2}\cdot\mathbf{v_3})}{(1-\hat{\mathbf{k}}\cdot\mathbf{v_2})(1-\hat{\mathbf{k}}\cdot\mathbf{v_3})}-\frac{(1-\mathbf{v_1}\cdot\mathbf{v_3})}{(1-\hat{\mathbf{k}}\cdot\mathbf{v_1})(1-\hat{\mathbf{k}}\cdot\mathbf{v_3})}-\frac{(1-\mathbf{v_2^2})}{(1-\hat{\mathbf{k}}\cdot\mathbf{v_2})^2}\right). \tag{18}$$

These results allow us to obtain the following conclusions. If the function A is greater than zero, then for ΔT small, we have destructive interference. That means that if the two accelerations occur in a small region, or if the particle has a large velocity, the radiated energy will be reduced because of the interference term. On the contrary, if $A<0$, for ΔT small we have constructive interference.

It is not easy to see whether the function A is going to be positive or negative. But from the Eq. (18) we can see that if $\mathbf{v_1}\parallel\mathbf{v_3}$, $\mathbf{v_1}\perp\mathbf{v_2}$, and $\mathbf{v_2}\perp\mathbf{v_3}$ it is more likely that the function A is going to be positive for the majority of the angles θ, ϕ. On the contrary, if $\mathbf{v_1}\perp\mathbf{v_3}$, it is more likely that the function A is going to be negative. Trajectories with such characteristics are depicted in Fig. 2.

It is interesting to analyze the case of a charge which is initially at rest, then it is accelerated to a velocity v_2 and finally it is stopped again. Then, with $\mathbf{v_1}=\mathbf{v_3}=0$ it is easy to see that for ΔT small enough, we have destructive interference, and for $\Delta T\to 0$ the radiated energy goes to zero. Usually this result is interpreted as follows. The electromagnetic radiation has a characteristic "formation time," which depends on the frequency of the radiation. If the particle is stopped after a time ΔT, no radiation is emitted for the frequencies which have a longer formation time than ΔT. In particular, if the particle is stopped right after it was accelerated ($\Delta T\to 0$), no radiation is emitted at all. By looking at the expression (16) we see that this interpretation is artificial because the interference term has periodicity. Then, if we

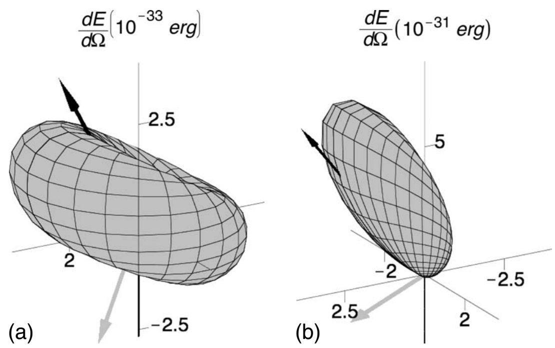

FIG. 3. Energy radiated by the charge. Left: $|\mathbf{v_1}|=0.2, |\mathbf{v_2}|=0.1$. Right: $|\mathbf{v_1}|=0.8, |\mathbf{v_2}|=0.7$. $\alpha=\pi/6$, $\Delta\tau=1$. The trajectory of the particle is indicated by the black line. The gray arrow indicates a vector in the direction of $\mathbf{v_2}-\mathbf{v_1}$.

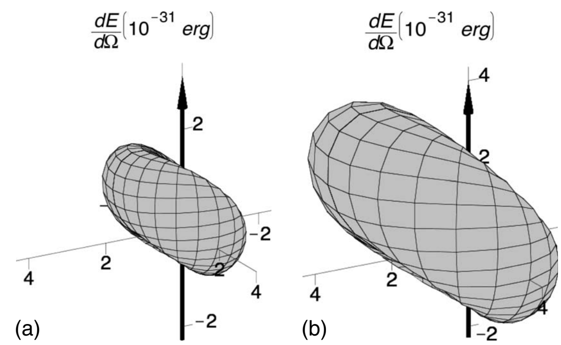

FIG. 4. Energy radiated by the double collision. Left: $\Delta T=0.2$. Right: $\Delta T=2$. $|\mathbf{v_1}|=0.3, |\mathbf{v_2}|=0.2, |\mathbf{v_3}|=0.1$ $\alpha_1=\pi/2$, $\alpha_2=0, \Delta\tau_1\approx\Delta\tau_2\approx 0.1$. The two collisions occur near the origin. The black line indicates the incoming and outgoing trajectories of the particle.

allow bigger values for ΔT, we will find a second minimum for the radiated energy for a given frequency, despite the formation time is already been satisfied. Moreover, as we have already seen, the interference term is not always destructive for small ΔT, its character depends on the initial and final velocities, and on the direction of the radiated waves.

Finally, it is worth discussing briefly the actual range of velocities in which our first-order approximation is valid. If one compares numerically the results of this approximation with the formulas found in [4] in the one-dimensional case, where no first-order approximation is been made, one can see that they give the same results in a wide range of initial and final velocities. It is necessary to go to the ultrarelativistic case to find differences. It is not difficult to show that even in cases of successive accelerations like the one proposed in the last paragraph, the approximation is successful for the range of velocities treated in the next section. We omit the numerical comparisons in order to keep brevity.

IV. EXAMPLES: RADIATION PATTERNS

A. Single acceleration

By using Eq. (14), it is possible to plot the radiated energy, using spherical coordinates with $\frac{dE}{d\Omega}$ as radius. The results are depicted in Fig. 3, where the time of the collision is taken as $\Delta\tau=1$. We can infer that with this value for $\Delta\tau$ and with the given values for the initial and final velocities, the proper acceleration experienced by the particle is $a_0 \approx \frac{|\mathbf{v2}|-|\mathbf{v1}|}{\Delta\tau}=0.1$. The results indicate that for velocities much smaller than the velocity of light, the radiated energy concentrates in directions perpendicular to the vector $\Delta\mathbf{v}=\mathbf{v_2}-\mathbf{v_1}$, as expected from the nonrelativistic analysis (see [3]). For relativistic velocities, the radiation is mainly emitted in the direction of the final velocity.

B. Successive accelerations

We consider the two cases given in Fig. 2. The radiated energy is now given by Eq. (17). The results for the first case are given in Fig. 4, where in the plot on the left the time between the collisions is small enough to observe interference effects, and in the figure on the right this time is great enough to make the interference effects unnoticeable. It is easy to see by comparing the two plots, that in the case of small ΔT, destructive interference occurs, as expected. The second case is shown in Fig. 5, where the two plots indicate again the radiation pattern with and without interference. This case is more interesting, because it is seen by comparing the two plots that if the time between the collisions is small, the interference term is constructive. It should be noted anyway that in this analysis, the time between the collisions must be much greater than the time that each collision lasts.

V. CONCLUSIONS

By using a first-order approximation in the proper time, an analytic expression for the radiation of a rapidly accelerated charge is found. This formula is easily applicable to successive accelerations. The model uses the formula of an instantaneously accelerated charge, introducing a cutoff in the radiated frequencies to take into account that the acceleration is finite. This cutoff is explicitly calculated as a function of the angle of radiation, and the initial and final velocities. It is found that the first-order approximation is well

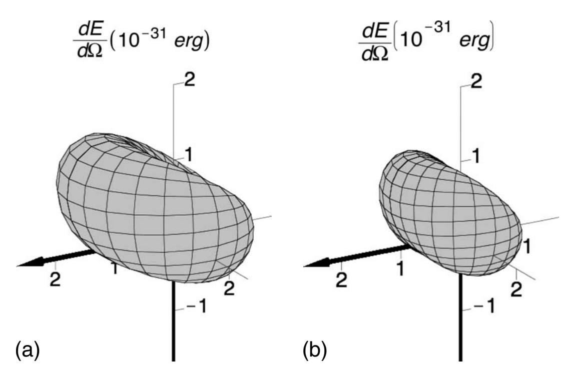

FIG. 5. Energy radiated by the double collision. Left: $\Delta T=0.2$. Right: $\Delta T=2$. $|\mathbf{v_1}|=0.3, |\mathbf{v_2}|=0.2, |\mathbf{v_3}|=0.1$ $\alpha_1=\pi/4$, $\alpha_2=\pi/2, \Delta\tau_1 \approx\Delta\tau_2\approx 0.1$. The two collisions occur near the origin. The black line indicates the incoming and outgoing trajectories of the particle.

suited if the velocities of the particle in the process are similar to the average velocity.

When two successive accelerations are taken into account, the interference term is calculated, and it is found that the interference can be constructive even if the time between the accelerations is short. The interference term has the form of the function sinc(x), and its sign depends on the angles and the velocities of the particle. It is not easy to see whether the interference is going to be constructive or destructive, but some paradigmatic cases can be found where the character of the interference is easy to predict.

In this analysis, it is shown that the total radiation in a process of successive accelerations is readily explained by means of the interference term, and that the concept of formation time of radiation is unnecessary in classical electrodynamics. This is because the reduction in radiation in some cases where the time between the collisions is short, is just a special case of an interference pattern.

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